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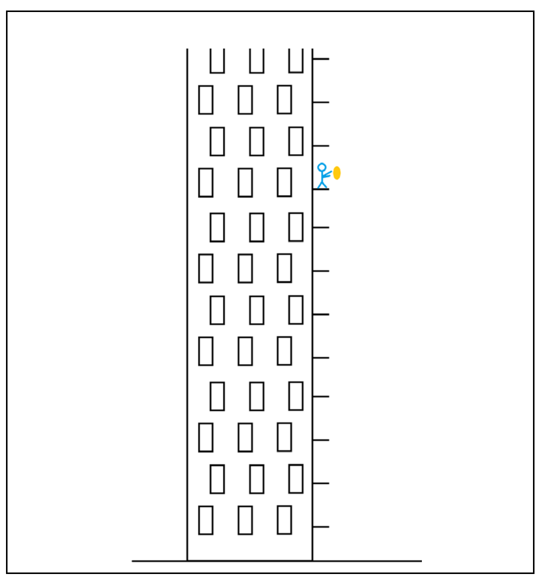
CURIOUS MATHEMATICS FOR FUN AND JOY

APRIL 2019

A SLIGHT VARIATION OF A CLASSIC PUZZLE: EGG DROPPING

Each floor of a certain exceptionally tall skyscraper has a balcony over which I can drop an egg and watch it hit the ground. The egg might or might not break upon landing. But, if an egg does break when it hits the ground at one level, I can be certain that another egg will break too if tossed from a higher level. If an egg does not break, I can be certain it will also not break if tossed from a lower level.



My job is to classify as many floors as possible from floor 1 consecutively up to some higher number floor, as either "egg breaking floors," which I'll label as **B** floors or as "non-egg-breaking floors," to be labelled **N** floors.

To do this, I have a fixed supply of eggs and I am willing to conduct a certain count of experiments. Each experiment consists of going to a floor of the skyscraper, tossing an egg over a balcony, and observing what happens. If the egg survives the toss, I can use it again for another experiment. But if it breaks, I am one egg down.

If I toss one egg from the first-floor balcony and it breaks, then I know every floor of the skyscraper is a B floor. If it does not break, then I've only succeeded in classifying only the first floor: it's an N floor.

So, with one egg and one experiment I can be sure to classify only floor 1.

With one egg and two experiments, I can be sure to classify the first two floors: drop the egg from the first floor and then from the second if it survives. The worst-case scenario is that I will ascertain that both floors are N floors and I will know no more. (Why can't I promise to classify floors 1, 2, and 3?)

a) Suppose I have one egg and am willing to run at most n experiments. Explain why I can promise to classify floors 1 through n, and promise no more.

b) I have two eggs and I am willing to up to conduct 4 experiments. Explain why I can promise to classify floors 1 through 10, and promise no more.

c) I have two eggs and I am willing to conduct up to n experiments. What is the largest count of floors I can promise to classify?

d) I have three eggs and I am willing to conduct up to n experiments. What is the largest count of floors I can promise to classify?

e) I have k eggs and I am willing to conduct up to n experiments. What is the largest count of floors I can promise to classify?

SOLVING THE PUZZLE

The egg-dropping puzzle is indeed a classic. But it is usually phrased this way:

Given a building with 100 floors and two eggs in hand, what is the least number of experiments you need to plan for to be sure you can classify each floor?

I was reminded of this puzzle when recently reading C. Brownell and S. Singh's brilliant new book *MATH RECESS* (Impress, 2019) It also appears in J. Konhauser, D. Velleman, and S. Wagon's gem *WHICH WAY DID THE BICYCLE GO?* (MAA, 1996).

I never took the time to figure out the puzzle myself, basically because I feared being lost in an unpleasant recursive relation.

But as I read Brownell and Sunil's discussion about using the puzzle in the classroom it occurred to me that fixing the number of experiments one is willing to perform is perhaps a more appealing entry into solving the puzzle. This essay illustrates my thoughts.

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ONE EXPERIMENT
If I am only willing to visit one floor and drop an egg, and I must classify the floors from floor 1 upwards, then I better toss that egg from floor 1. (If a drop an egg from a higher floor and it breaks, then I know nothing about floor 1.)

With one experiment I can only promise to classify floor 1.

ONE EGG
With 1 egg and 1 experiment I can promise to classify floor 1.

With 1 egg and up to 2 experiments I can promise to classify floors 1 and 2. Here's how:

Drop an egg from floor 1. If it breaks, I know the classification of all floors. If it doesn't, then next toss it from floor 2. I can't run any more experiments, but I will know at the very least the labels of floors 1 and 2.

I can't promise to classify floors 1, 2, and 3. Here's why:

If I first drop the egg from above floor 1 and it breaks, I am done and I won't know anything about floor 1 itself. So, my first experiment must be from floor 1.

If the egg doesn't break from experiment 1, I can drop the egg again. If I do so from a floor higher than floor 2 and it breaks, then I won't know anything about floor 2. So, my second experiment must be from floor 2.

If the egg doesn't break from experiment 2, then I will have classified floors 1 and 2, but know nothing about floors 3 and higher. I am out of experiments can't go any further.

But we can see with 1 egg and 3 experiments, a third experiment will have to be conducted from floor 3 and I can promise to classify floors 1, 2, and 3.

A fourth experiment will have to be from floor 4, and I will be able to promise to classify floors 1, 2, 3, and 4.

And so on.

We conclude: *With one egg and a run of at most n experiments I can promise to classify floors 1, 2, 3,…, n (and can't promise any more).*

k EGGS, n EXPERIMENTS
Let $E_k(n)$ be the maximal number of floors

$$1, 2, 3, \ldots, E_k(n)$$

I can promise to classify with k eggs in hand when I am willing to run up to n experiments.

The previous two sections of work show

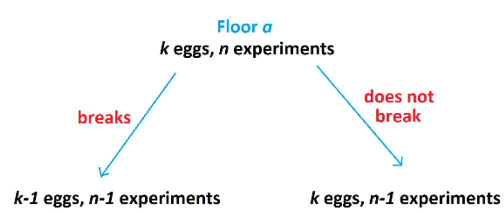
$$E_k(1) = 1 \text{ for all } k$$

(if I am willing to run only one experiment, having k eggs in hand is equivalent to having just 1 egg in hand), and

$$E_1(n) = n \ .$$

Suppose I have k eggs and am willing to run up to n experiments.

Let's start by dropping an egg from floor a.



If the egg breaks, then we know nothing about floors 1, 2, …., $a-1$ but will have classified all higher floors as B floors. We still need to classify the lower floors and have $k-1$ eggs and $n-1$ experiments to

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do it. By definition, $E_{k-1}(n-1)$ is the biggest number of floors we can classify in this situation, so for optimal results we should have $a = E_{k-1}(n-1)+1$. We will then have classified all floors of the building.

If the egg does not break, then we know floors 1, 2, …, a are N floors, and we still have k eggs and $n-1$ experiments to test more floors. We know, for sure, we can promise to classify $E_k(n-1)$ more floors.

So, either way, we can promise to classify floors

1, 2, 3, …, $E_{k-1}(n-1)$, $E_{k-1}(n-1)+1$, …, $E_{k-1}(n-1)+1+E_k(n-1)$

and this is the best we can promise. We have then

$$E_k(n) = E_{k-1}(n-1)+E_k(n-1)+1.$$

This is great! It's a recursion relation that allows us to complete an entire table of values. The numbers in blue are the initial values we computed with one egg and with one experiment. Each interior entry is one more than the sum of a pair of numbers above it.

k eggs

n experiments / $E_k(n)$	1	2	3	4	5	6	7	8
1	1	1	1	1	1	1	1	1
2	2	3	3	3	3	3	3	3
3	3	6	7	7	7	7	7	7
4	4	10	14	15	15	15	15	15
5	5	15	25	30	31	31	31	31
6	6	21	41	56	62	63	63	63
7	7	28	63	98	119	126	127	127

6 + 7 + 1
= 14

And as a reminder, each entry in the table is the highest floor, starting with floor 1 and working our way up, we can promise to classify with k eggs and up to n experiments.

CHALLENGE: Suppose you have 3 eggs in hand and are willing to conduct at most 6 experiments. From the table we see that $E_3(6) = 41$ so you can promise to classify floors 1, 2, 3, …, 41.

But what is your strategy?

Our reasoning shows that you should first throw an egg from floor $E_2(5)+1 = 16$.

If it breaks, then we are left with classifying 15 floors with 2 eggs and up to 5 experiments.

If it does not break, then with 3 eggs in hand and up to 5 more experiments to run we should next go to floor $16 + (E_2(4)+1) = 16+10+1 = 27$.

Your challenge: *Make a complete map as how exactly to classify each of these 41 floors.*

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STRUCTURE IN THE TABLE

The numbers in the first column are the counting numbers.

Puzzle: *Given a building with 100 floors and one egg in hand, what is the least number of experiments you need plan for to be sure you will be able to classify each floor?*

Answer: Looking at the first column you might well need to conduct possibly 100 experiments. (This will happen if they all turn out to be N floors.)

In general, if there are f floors, you will need to plan for possibly f experiments.

The numbers in the second column are the triangular numbers. This follows since the recursion relation here reads

$$\begin{aligned} E_2(n) &= E_1(n-1) + E_2(n-1) + 1 \\ &= n - 1 + E_2(n-1) + 1 \\ &= E_2(n-1) + n. \end{aligned}$$

From $E_2(1) = 1$ we get

$$E_2(2) = 1 + 2$$
$$E_2(3) = 1 + 2 + 3$$
$$E_2(4) = 1 + 2 + 3 + 4$$

and so on. We know the formula for the n-th triangular number:

$$E_2(n) = \frac{n(n+1)}{2}.$$

The Classic Puzzle: *Given a building with 100 floors and two eggs in hand, what is the least number of experiments you need plan for to be sure you will be able to classify each floor?*

Answer: Since $E_2(13) = 91$ and $E_2(14) = 105$, we may need to conduct as many as 14 experiments.

In general, if there are f floors, the minimal number of experiments we need to plan for is the smallest n satisfying

$$\frac{n(n+1)}{2} \geq f.$$

That is, we need $n^2 + n \geq 2f$ or

$$\left(n + \frac{1}{2}\right)^2 \geq 2f + \frac{1}{4}$$

showing that

$$n = \left\lceil \frac{\sqrt{8f+1} - 1}{2} \right\rceil$$

(where the brackets mean to round up to the nearest integer).

The numbers in the higher columns seem a little more mysterious. We'll attend to those at the end the of the essay.

Looking at the rows, it seems that values stabilize to the right of the diagonal entries $E_n(n)$. And this makes sense: if you are willing to conduct n experiments, then you'll use at most n eggs, and so having more than this many eggs in hand is irrelevant.

$$E_n(n) = E_{n+1}(n) = E_{n+2}(n) = \cdots,$$

That is,

$$E_k(n) = E_n(n) \text{ for } k \geq n.$$

And look at these "stabilized" values:

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$$E_1(1) = 1$$
$$E_2(2) = 3$$
$$E_3(3) = 7$$
$$E_4(4) = 15$$
$$E_5(5) = 31$$

and so on.

Is $E_n(n) = 2^n - 1$?
The answer is yes!

If I have a $k = 1,000,000$ eggs, then

$$E_k(n) = E_{k-1}(n-1) + E_k(n-1) + 1$$

reads as

$$E_n(n) = E_{n-1}(n-1) + E_{n-1}(n-1) + 1$$

for $n = 1, 2, 3, \ldots, 999999$ showing that each "stabilization" number is double the previous one plus 1, at least for a good start. As $E_1(1) = 1$, this is precisely the recursive relation that sets $E_n(n) = 2^n - 1$.

$$E_1(1) = 1$$
$$E_2(2) = 2 \cdot 1 + 1 = 3$$
$$E_3(3) = 2 \cdot 3 + 1 = 7$$

and if we know $E_k(k) = 2^k - 1$, then
$E_{k+1}(k+1) = 2 \cdot (2^k - 1) + 1 = 2^{k+1} - 1$.

So $E_n(n) = 2^n - 1$ for the first $999,999$ values. But since choosing $k = 1,000,000$ was arbitrary, we can deduce $E_n(n) = 2^n - 1$ for all values.

A Classic Variation: *Given a building with 100 floors and an unlimited supply of eggs, what is the least number of experiments you need plan for to be sure you will be able to classify each floor?*

Answer: Looking at the table of $E_k(N)$ values, we see that no entry in the first six rows has value 100 or greater. With 7 experiments we can classify 119 floors (using 5 eggs) and so this is the minimum number experiments you need to plan for.

In general, if there are f floors, the minimal number of experiments we need to plan for is the smallest n satisfying

$$2^n - 1 \geq f.$$

That is, we need $n \geq \log_2(f+1)$ showing that

$$n = \lceil \log_2(f+1) \rceil.$$

A GENERAL FORMULA FOR $E_k(n)$

The "$+1$" in the recursion formula for $E_k(n)$ just feels irksome to my sensibilities. One way to obviate such annoyances is to subtract two copies of the same recursion relation.

Set $D_k(n) = E_{k+1}(n) - E_k(n)$.

Subtraction then shows

$$D_k(n) = D_{k-1}(n-1) + D_k(n-1).$$

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subtract

$$\begin{array}{lllll} E_{k+1}(n) & = E_k(n-1) & + & E_{k+1}(n-1) & +1 \\ E_k(n) & = E_{k-1}(n-1) & + & E_k(n-1) & +1 \\ \hline D_k(n) & = D_{k-1}(n-1) & + & D_k(n-1) & \end{array}$$

This looks like a Pascal-triangle type of recursion relation: each entry is the direct sum of the two entries above it.

In fact, if we make a table of the values of $D_k(n)$ (just taking differences of the $E_k(n)$ values), look at what we get!

	$D_k(n)$	k: 1	2	3	4	5
n	1	0	0	0	0	0
	2	1	0	0	0	0
	3	3	1	0	0	0
	4	6	4	1	0	0
	5	10	10	5	1	0

Whoa! It looks like we're getting a right portion of Pascal's triangle. We conjecture

$$D_k(n) = \binom{n}{k+1}.$$

(Double check my indices here!)

Since the first few rows of the table match this formula and also match Pascal's triangle, and both $D_k(n)$ and the entries of Pascal's triangle satisfy the same recursion relation, the table is sure to continue to match entries of Pascal's triangle! This formula for $D_k(n)$ is thus indeed always valid.

And this means we have a formula for $E_k(n)$.

$$\begin{aligned} E_k(n) = {} & E_k(n) - E_{k-1}(n) \\ & + E_{k-1}(n) - E_{k-2}(n) \\ & + \cdots \\ & + E_2(n) - E_1(n) \\ & + E_1(n) \end{aligned}$$

$$= D_{k-1}(n) + D_{k-2}(n) + \cdots D_1(n) + n$$

$$= \binom{n}{k} + \binom{n}{k-1} + \cdots + \binom{n}{2} + n$$

$$= \binom{n}{k} + \binom{n}{k-1} + \cdots + \binom{n}{2} + \binom{n}{1}$$

For example,

$$\begin{aligned} E_3(4) &= \binom{4}{3} + \binom{4}{2} + \binom{4}{1} \\ &= 4 + 6 + 4 \\ &= 14 \end{aligned}$$

which is correct!

$E_k(n)$ is the sum of k entries in the nth row of Pascal's triangle.

Double whoa!

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RESEARCH CORNER

I have a supply of chicken eggs and a supply of emu eggs. Emu eggs are harder to break than chicken eggs. Thus by dropping eggs off the balconies of a building I can classify the floors of a building as either:

> **N** for non-egg-breaking
> **C** for just chicken-egg-breaking (but not emu-egg breaking)
> **B** for both emu and chicken-egg-breaking.

An "experiment" consists of going to a floor of the building and dropping just one egg—either a chicken egg or an emu egg—from that floor.

My job is to classify the floors from floor 1 consecutively up to some higher number floor.

Suppose I have a supply of c chicken eggs, e emu eggs, and I am willing to run up to n experiments. Let $E(n,c,e)$ be the highest floor number I can promise to classify (from floor 1 up to this floor number).

Study the values of $E(n,c,e)$.

Is there a relationship between the value of $E(n,c,e)$ and the values of $E_c(n)$ and $E_e(n)$?

(Of course, we can assume that if throwing an emu egg from a floor breaks the emu egg, then a chicken egg thrown from that floor is sure to break as well. And if a chicken egg survives being thrown from a certain floor, an emu egg will as well.)

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