Finding Square Roots Using Newton's Method

Let $A > 0$ be a positive real number. We want to show that there is a real number x with $x^2 = A$. We already know that for many real numbers, such as $A = 2$, there is no rational number x with this property. Formally, let $fx) := x^2 - A$. We want to solve the equation $f(x) = 0$.

Newton gave a useful general recipe for solving equations of the form $f(x) = 0$. Say we have some approximation x_k to a solution. He showed how to get a better approximation x_{k+1}. It works most of the time if your approximation is close enough to the solution.

Here's the procedure. Go to the point $(x_k,\, f(x_k))$ and find the tangent line. Its equation is

$$y = f(x_k) + f'(x_k)(x - x_k).$$

The next approximation, x_{k+1}, is where this tangent line crosses the x axis. Thus,

$$0 = f(x_k) + f'(x_k)(x_{k+1} - x_k), \qquad \text{that is,} \qquad x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}.$$

Applied to compute square roots, so $f(x) := x^2 - A$, this gives

$$x_{k+1} = \frac{1}{2}\left(x_k + \frac{A}{x_k}\right). \tag{1}$$

From this, by simple algebra we find that

$$x_{k+1} - x_k = \frac{1}{2x_k}(A - x_k^2). \tag{2}$$

Pick some x_0 so that $x_0^2 > A$. then equation (2) above shows that subsequent approximations x_1, x_2, ..., are monotone decreasing. Equation (2) then shows that the sequence $x_1 \ge x_2 \ge x_3 \ge \ldots$, is monotone decreasing and non-negative. By the monotone convergence property, it thus converges to some limit x.

I claim that $x^2 = A$. Rewrite (2) as $A - x_k^2 = 2x_k(x_{k+1} - x_k)$ and let $k \to \infty$. Since $x_{k+1} - x_k \to 0$ and x_k is bounded, this is obvious.

We now know that $\sqrt{A}$ exists as a real number. then it is simple to use (1) to verify that

$$x_{k+1} - \sqrt{A} = \frac{1}{2x_k}(x_k - \sqrt{A})^2. \tag{3}$$

Equation (3) measures the error $x_{k+1} - \sqrt{A}$. It shows that the error at the next step is the *square* of the error in the previous step. Thus, if the error at some step is roughly 10^{-6} (so 6 decimal places), then at the next step the error is roughly 10^{-12} (so 12 decimal places).

Example: To 20 decimal places, $\sqrt{7} = 2.64575131106459059050$. Let's see what Newton's method gives with the initial approximation $x_0 = 3$:

$$x_1 = 2.66666666666666666666 \qquad x_2 = 2.64583333333333333333$$
$$x_3 = 2.64575131233595800525 \qquad x_4 = 2.64575131106459059085$$

Remarkable accuracy.